

# Reflected BSDE with quadratic growth and unbounded terminal value

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## Abstract

In this paper we prove the existence of a solution for reflected BSDE's whose coefficient is of quadratic growth in  $z$  and of linear growth in  $y$ , with an unbounded terminal value.

**Keywords:** Reflected Stochastic Differential Equations, reflected ordinary differential equation, characterization of the solution, quadratic growth

## 1 Introduction

In this paper we are interested with the following real valued reflected backward stochastic differential equations (RBSDE's in short) with one continuous barrier

$$\begin{aligned} Y_t &= \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s + K_T - K_t, 0 \leq t \leq T \\ Y_t &\geq L_t, 0 \leq t \leq T, \int_0^T (Y_s - L_s) dK_s = 0 \end{aligned}$$

where  $(B_t)$  is a standard Brownian motion. In our setting the coefficient, namely  $f$ , is of quadratic growth in  $z$  and of linear growth in  $y$ .

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In 1996, El Karoui et al. [4] first introduced this kind of equations and proved the existence and uniqueness of the solution under a Lipschitz condition in  $y$  and  $z$ . Then in 1997 Matoussi [11] studied the case when  $f$  is of linear growth in  $y$  and  $z$ . When the terminal value  $\xi$  is square integrable he proved the existence of a maximal and a minimal solution. Later RBSDE's, whose coefficients are quadratic growth in  $z$ , have been studied by Kobylanski, Lepeltier, Quenez, Torres in [7], but they required the terminal value  $\xi$  is bounded.

In an interesting paper, Briand and Hu [2] relaxed the boundness of  $\xi$  for non reflected BSDE's whose coefficients is quadratic growth in  $z$ . In this work we use a similar approach in the case of RBSDE's, with the help of existence results contained in [7].

The next section is devoted to the assumptions and the claim of the main result theorem 2.1. The third section gives some estimation results which are important to establish the proof of theorem 2.1 in section 4. Then section 5 is devoted to get an extension to the case that  $f$  is superlinear in  $y$ . Finally in section 6 (Appendix) we study the existence, uniqueness and characterization of the solution for backward ordinary differential equations with one lower continuous barrier, which is a key point in the technics used in section 3.

## 2 Assumptions and Main result

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space, and  $(B_t)_{0 \leq t \leq T} = (B_t^1, B_t^2, \dots, B_t^d)'_{0 \leq t \leq T}$  be a  $d$ -dimensional Brownian motion defined on a finite interval  $[0, T]$ ,  $0 < T < +\infty$ . Denote by  $\{\mathcal{F}_t; 0 \leq t \leq T\}$  the standard filtration generated by the Brownian motion  $B$ , i.e.  $\mathcal{F}_t$  is the completion of

$$\mathcal{F}_t = \sigma\{B_s; 0 \leq s \leq t\},$$

with respect to  $(\mathcal{F}, P)$ . We denote by  $\mathcal{P}$  the  $\sigma$ -algebra of predictable sets on  $[0, T] \times \Omega$ .

We shall need the following spaces:

$$\begin{aligned} \mathbf{L}^2(\mathcal{F}_t) &= \{\eta : \mathcal{F}_t\text{-measurable random real-valued variable, s.t. } E(|\eta|^2) < +\infty\}, \\ \mathbf{H}_n^2(0, T) &= \{(\psi_t)_{0 \leq t \leq T} : \text{predictable process valued in } \mathbb{R}^n, \text{ s.t. } E \int_0^T |\psi(t)|^2 dt < +\infty\}, \\ \mathbf{S}^2(0, T) &= \{(\psi_t)_{0 \leq t \leq T} : \text{progressively measurable real-valued process,} \\ &\quad \text{s.t. } E(\sup_{0 \leq t \leq T} |\psi(t)|^2) < +\infty\}, \\ \mathbf{A}^2(0, T) &= \{(K_t)_{0 \leq t \leq T} : \text{adapted continuous increasing process,} \\ &\quad \text{s.t. } K(0) = 0, E(K(T)^2) < +\infty\}. \end{aligned}$$

$\mathbf{S}^\infty(0, T)$  denotes the set of predictable bounded processes.

In this paper, we work under the following assumptions:

**Assumption 2.1.** a coefficient  $f : [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ , is linear increasing in  $y$  and quadratic growth in  $z$ : there exists  $\alpha, \beta \geq 0, \gamma > 0$ , satisfying  $\alpha \geq \frac{\beta}{\gamma}$ , such that for  $\forall(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$ ,

$$|f(t, y, z)| \leq \alpha + \beta |y| + \frac{\gamma}{2} |z|^2; \quad (1)$$

moreover  $f(t, y, z)$  is continuous in  $(y, z)$ , for all  $t \in [0, T]$ .

**Assumption 2.2.** a terminal condition  $\xi \in \mathbf{L}^2(\mathcal{F}_T)$ , such that

$$E[e^{\gamma e^{\beta T} |\xi|}] < +\infty.$$

**Assumption 2.3.** a barrier  $L$ , which is a bounded continuous process, with  $L_T \leq \xi$ , and for  $\forall t \in [0, T]$ ,  $|L_t| \leq a_t$ , where  $a_t$  is a deterministic and continuous process.

For the terminal condition, we propose another stronger assumption:

**Assumption 2.4** a terminal time  $\xi \in \mathbf{L}^2(\mathcal{F}_T)$ , such that  $E[e^{2\gamma e^{\beta T} |\xi|}] < +\infty$ .

**Remark 2.1.** From the assumption 2.3, we know that  $\xi$  has a lower bound in view of  $\xi \geq L_T \geq -a_T$ .

Our main result in this paper is:

**Theorem 2.1.** Under the assumptions 2.1-2.3, the reflected BSDE associated to  $(\xi, f, L)$  admits at least a solution, i.e. there exists a triplet  $(Y_t, Z_t, K_t)_{0 \leq t \leq T}$ , with  $Y \in \mathbf{S}^2(0, T)$ , and  $K \in \mathbf{A}^2(0, T)$ , such that

$$\begin{aligned} Y_t &= \xi + \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dB_s, \\ Y_t &\geq L_t, \quad \int_0^T (Y_t - L_t) dK_t = 0. \end{aligned}$$

Moreover if assumption 2.4 holds, then  $Z \in \mathbf{H}_d^2(0, T)$ .

### 3 Estimation results

To prove theorem 2.1, we need prove an estimation result. Define

$$\mathbf{L}_\gamma^2(\mathbb{R}) = \{ (v_t)_{0 \leq t \leq T} : [0, T] \rightarrow \mathbb{R}, \text{ s.t. } \int_0^T e^{\gamma t} |v_t|^2 dt < \infty \}, \text{ for } \gamma \in \mathbb{R}.$$

**Lemma 3.1.** Let assumption 2.1 hold and  $\xi$  be a bounded  $\mathcal{F}_T$ -measurable random variable. If  $(Y_t, Z_t, K_t)_{0 \leq t \leq T}$  is a solution of the RBSDE  $(\xi, f, a)$  in  $\mathbf{S}^\infty(0, T) \times \mathbf{H}_d^2(0, T) \times \mathbf{A}^2(0, T)$ , then

$$L_t \leq Y_t \leq \frac{1}{\gamma} \ln(E[\theta_t(\xi) | \mathcal{F}_t]).$$

Here the mapping  $\theta_t(\cdot) : \mathbf{R} \rightarrow \mathbf{L}_\gamma^2(\mathbf{R})$  is defined that for  $x \in \mathbf{R}$ ,  $(\theta_t(x), k_t(x))$  is the unique solution of following reflected backward ordinary differential equation,

$$\begin{aligned} \theta_t(x) &= e^{\gamma x} + \int_t^T H(\theta_s(x)) ds + k_T(x) - k_t(x), \\ \theta_t(x) &\geq e^{\gamma a_t}, \quad \int_0^T (\theta_t(x) - e^{\gamma a_t}) dk_t(x) = 0. \end{aligned} \tag{2}$$

with  $H(p) = p(\alpha\gamma + \beta \ln p)1_{[1, +\infty)}(p) + \gamma\alpha 1_{(-\infty, 1)}(p)$ .

*Proof.* Consider the change of variable

$$P_t = e^{\gamma Y_t}, Q_t = \gamma e^{\gamma Y_t} Z_t = \gamma P_t Z_t, J_t = \int_0^t \gamma e^{\gamma Y_s} dK_s.$$

It is easy to check that  $(Y, Z, K)$  is a solution of the RBSDE $(\xi, f, L)$  if and only if  $(P, Q, J)$  is a solution of the RBSDE $(e^{\gamma \xi}, F, e^{\gamma L_t})$ , where

$$F(s, p, q) = 1_{\{p > 0\}}(\gamma p f(s, \frac{\ln p}{\gamma}, \frac{q}{\gamma p}) - \frac{1}{2} \frac{|q|^2}{p}),$$

i.e. the triplet  $(P_t, Q_t, J_t)_{0 \leq t \leq T}$  satisfies

$$\begin{aligned} P_t &= e^{\gamma \xi} + \int_t^T F(s, P_s, Q_s) ds + J_T - J_t - \int_t^T Q_s dB_s, \\ P_t &\geq e^{\gamma L_t}, \int_0^T (P_t - e^{\gamma L_t}) dJ_t = 0. \end{aligned}$$

Then in order to get the integral property of  $Y$ , it is sufficient to study the integrability of the process  $P$ . First  $P_t \geq e^{\gamma L_t}$ , then it remains to find out an upper bound of  $P$ .

We define the mapping  $\theta_t(\cdot) : \mathbf{R} \rightarrow \mathbf{L}_\gamma^2(\mathbf{R})$ , for  $x \in \mathbf{R}$ ,  $\theta_t(x)$  with an increasing process  $k_t(x)$ , is a unique solution of the reflected BODE with coefficient  $H$ , deterministic barrier  $e^{\gamma a_t}$ , and terminal condition  $e^{\gamma x} \in \mathbb{R}$ , satisfying  $x \geq a_T$ ; i.e. (2) is satisfied. Thanks to theorem 6.2 in the Appendix, we know that  $\theta_t(x)$  exists and can be written in the following forms

$$\begin{aligned} \theta_t(x) &= \sup_{t \leq s \leq T} \varphi_t(s, e^{\gamma a_s} 1_{\{s < T\}} + e^{\gamma x} 1_{\{s = T\}}) \\ &= \max\{\varphi_t(T, e^{\gamma x}), \sup_{t \leq s < T} \varphi_t(s, e^{\gamma a_s})\} \\ &= \sup_{t \leq s \leq T} [\int_t^s H(\theta_r(x)) dr + e^{\gamma a_s} 1_{\{s < T\}} + e^{\gamma x} 1_{\{s = T\}}], \end{aligned}$$

where  $\varphi_t(s, e^{\gamma a_s} 1_{\{s < T\}} + e^{\gamma x} 1_{\{s = T\}})$  is the solution of the non-reflected BODE on  $[0, s]$  with coefficient  $H$  and terminal value  $e^{\gamma a_s} 1_{\{s < T\}} + e^{\gamma x} 1_{\{s = T\}}$ , i.e. the followings hold

$$\begin{aligned} \varphi_t(T, e^{\gamma x}) &= e^{\gamma x} + \int_t^T H(\varphi_r(T, e^{\gamma x})) dr, \\ \varphi_t(s, e^{\gamma a_s}) &= e^{\gamma a_s} + \int_t^s H(\varphi_r(s, e^{\gamma a_s})) dr, \text{ for } 0 \leq s < T. \end{aligned} \tag{3}$$

For a bounded  $\mathcal{F}_T$ -measurable random variable  $\xi$ , we get

$$\theta_t(\xi) = \max\{\varphi_t(T, e^{\gamma \xi}), \sup_{t \leq s < T} \varphi_t(s, e^{\gamma a_s})\},$$

which is also an  $\mathcal{F}_T$ -measurable random variable. Since

$$\theta_t(\xi) = \sup_{t \leq s \leq T} \left[ \int_t^s H(\theta_r(\xi)) dr + e^{\gamma a_s} 1_{\{s < T\}} + e^{\gamma \xi} 1_{\{s = T\}} \right],$$

for any stopping time  $\tau$ , such that  $t \leq \tau \leq T$ , we have

$$\theta_t(\xi) \geq \int_t^\tau H(\theta_r(\xi)) dr + e^{\gamma a_\tau} 1_{\{\tau < T\}} + e^{\gamma \xi} 1_{\{\tau = T\}}.$$

So

$$\theta_t(\xi) \geq \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} \int_t^\tau H(\theta_r(\xi)) dr + e^{\gamma a_\tau} 1_{\{\tau < T\}} + e^{\gamma \xi} 1_{\{\tau = T\}},$$

where  $\mathcal{T}_{t,T}$  is the set of the stopping times valued in  $[t, T]$ .

Denote  $\Theta_t(\xi) := E[\theta_t(\xi) | \mathcal{F}_t]$ , then we have

$$\begin{aligned} \Theta_t(\xi) &\geq \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} E \left[ \int_t^\tau H(\theta_r(\xi)) dr + e^{\gamma a_\tau} 1_{\{\tau < T\}} + e^{\gamma \xi} 1_{\{\tau = T\}} \middle| \mathcal{F}_t \right] \\ &\geq \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} E \left[ \int_t^\tau E[H(\theta_r(\xi)) | \mathcal{F}_r] dr + e^{\gamma a_\tau} 1_{\{\tau < T\}} + e^{\gamma \xi} 1_{\{\tau = T\}} \middle| \mathcal{F}_t \right]. \end{aligned}$$

Set  $\Phi_t(\xi)$  equal to the right side; by the optimal stopping problem, we know that there exist  $(\Psi(\xi), \Lambda(\xi)) \in \mathbf{H}_d^2(0, T) \times \mathbf{A}^2(0, T)$ , such that  $(\Phi(\xi), \Psi(\xi), \Lambda(\xi))$  is the solution of the RBSDE( $e^{\gamma \xi}, E[H(\theta_t(\xi)) | \mathcal{F}_t], e^{\gamma a}$ ).

From assumption 2.1, it follows that the function  $H$  is convex, increasing in  $p$ . And  $F(s, p, q) \leq H(p)$ , for any  $s \in [0, T]$ ,  $p \in \mathbb{R}$ ,  $q \in \mathbb{R}^d$ . So for  $r \in [0, T]$ , we have

$$\begin{aligned} E[H(\theta_r(\xi)) | \mathcal{F}_r] &\geq H(E[\theta_r(\xi) | \mathcal{F}_r]) = H(\Theta_r(\xi)) \\ &\geq H(\Phi_r(\xi)) \geq F(r, \Phi_r(\xi), \Psi_r(\xi)). \end{aligned}$$

Since  $\xi$  is a bounded  $\mathcal{F}_T$ -measurable random variable, it follows that  $\Phi_t(\xi)$  and  $P_t$  are bounded. Since  $H(p)$  is locally Lipschitz, we can apply the trajectory comparison theorem for these RBSDEs, and get for  $t \in [0, T]$ ,

$$\Theta_t(\xi) \geq \Phi_t(\xi) \geq P_t.$$

Consequently

$$Y_t \leq \frac{1}{\gamma} \ln \Theta_t(\xi) = \frac{1}{\gamma} \ln(E[\theta_t(\xi) | \mathcal{F}_t]).$$

□

**Lemma 3.2.** *Let assumptions 2.1 and 2.3 hold, and  $\xi$  be a  $\mathcal{F}_T$ -measurable bounded random variable. If  $(Y_t, Z_t, K_t)_{0 \leq t \leq T}$  is a solution of the RBSDE( $\xi, f, L$ ), then*

$$L_t \leq Y_t \leq \frac{1}{\gamma} \ln(E[\theta_t(\xi \vee a_T) | \mathcal{F}_t]). \quad (4)$$

*Proof.* Obviously  $Y_t \geq L_t$ . For the right side, consider the RBSDE( $\xi \vee a_T, f, a$ ); since  $a$  is a bounded continuous process, by [7], it admits a maximal solution  $(Y^a, Z^a, K^a)$ . From the comparison theorem, we have  $Y_t \leq Y_t^a$ . Thanks to lemma 3.1,  $Y_t^a \leq \frac{1}{\gamma} \ln(E[\theta_t(\xi \vee a_T)|\mathcal{F}_t])$ , which follows

$$Y_t \leq \frac{1}{\gamma} \ln(E[\theta_t(\xi \vee a_T)|\mathcal{F}_t]).$$

□

**Remark 3.1.** We can also get some comparison results of  $\theta_t(x)$ . Recalling the results in [2], we can solve equations (3) explicitly. From their forms, it is easy to check that  $\varphi_t(T, e^{\gamma x})$  and  $\varphi_t(s, e^{\gamma a_s})$  are decreasing in  $t$ , and  $\varphi_t(T, e^{\gamma x})$  is increasing and continuous in  $x$ . So  $\theta_t(x)$  is increasing in  $x$ .

For  $t_1, t_2 \in [0, T]$ , with  $t_1 \leq t_2$ , we have

$$\varphi_{t_1}(T, e^{\gamma x}) \geq \varphi_{t_2}(T, e^{\gamma x}) \text{ and } \varphi_{t_1}(s, e^{\gamma a_s}) \geq \varphi_{t_2}(s, e^{\gamma a_s}).$$

Remember that

$$\begin{aligned} \theta_{t_1}(x) &= \max\{\varphi_{t_1}(T, e^{\gamma x}), \sup_{t_1 \leq s \leq t_2} \varphi_{t_1}(s, e^{\gamma a_s}), \sup_{t_2 \leq s \leq T} \varphi_{t_1}(s, e^{\gamma a_s})\}, \\ \theta_{t_2}(x) &= \max\{\varphi_{t_2}(T, e^{\gamma x}), \sup_{t_2 \leq s \leq T} \varphi_{t_2}(s, e^{\gamma a_s})\}, \end{aligned}$$

then we obtain  $\theta_{t_1}(x) \geq \theta_{t_2}(x)$ , i.e.  $\theta_t(x)$  is decreasing in  $t$ .

## 4 The proof of theorem 2.1

Now we can prove our main result. Before beginning the proof, we present a monotone stability theorem, which is proved in theorem 4 of [7].

**Theorem 4.1.** Let  $(\xi^p)_{p \in \mathbb{N}}$ ,  $\xi$  be a family of terminal condition,  $(g^p)_{p \in \mathbb{N}}$ ,  $g$  be a family of coefficients,  $L$  be a continuous bounded process, which satisfy:

(a) there exists a constant  $b > 0$ , such that for each  $p$ ,  $|\xi^p| \leq b$ , and  $|L_t| \leq b$ , for  $t \in [0, T]$ , with  $\xi^p \geq L_T$ .

(b)  $g^p, g : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  are  $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable, and there exists a function  $l_1$  of the form  $l_1(y) = a_1(1 + |y|)$ , with  $a_1 > 0$ , and a constant  $A$ , such that for each  $p$ ,

$$|g^p(t, y, z)| \leq l_1(y) + A|z|^2 \text{ and } |g(t, y, z)| \leq l_1(y) + A|z|^2.$$

(c) the sequence  $(g^p)$  converge increasingly (resp. decreasingly) to  $g$  locally uniformly on  $[0, T] \times \mathbb{R} \times \mathbb{R}^d$ , and  $(\xi^p)$  converge increasingly (resp. decreasingly) to  $\xi$ .

For each  $p$ , let  $(Y^p, Z^p, K^p)$  be the maximal solution of the RBSDE( $\xi^p, g^p, L$ ). Then the sequence  $(Y^p)$  converges increasingly (resp. decreasingly) to  $Y$  uniformly on  $[0, T]$ ,  $(Z^p)$  converges to  $Z$  in  $\mathbf{H}_d^2(0, T)$ , and  $(K^p)$  converges decreasingly (resp. increasingly) to  $K$  uniformly on  $[0, T]$ , where  $(Y, Z, K)$  is the maximal solution of the RBSDE( $\xi, g, L$ ).

**Remark 4.1.** The results still hold if we consider the minimal solutions of the RBSDEs.

**Proof of theorem 2.1:**

By remark 2.1, we know that  $\xi$  has a lower bound. So we only need to consider the approximation of the upper side. For  $n \geq a_T$ , we set  $\xi^n := \xi \wedge n$ . It is known from [7] that there exists a maximal bounded solution  $(Y^n, Z^n, K^n)$  to the RBSDE $(\xi^n, f, L)$ ,

$$\begin{aligned} Y_t^n &= \xi^n + \int_t^T f(s, Y_s^n, Z_s^n) ds + K_T^n - K_t^n - \int_t^T Z_s^n dB_s, \\ Y_t^n &\geq L_t, \int_0^T (Y_t^n - L_t) dK_t^n = 0. \end{aligned}$$

Here  $(Y^n, Z^n, K^n) \in \mathbf{S}^2(0, T) \times \mathbf{H}_d^2(0, T) \times \mathbf{A}^2(0, T)$ . Then from lemma 3.2, we get

$$L_t \leq Y_t^n \leq \frac{1}{\gamma} \ln(E[\theta_t(\xi^n \vee a_T) | \mathcal{F}_t]).$$

By the comparison theorem under superlinear condition in the Appendix of [13], it follows that for  $t \in [0, T]$ ,

$$Y_t^n \leq Y_t^{n+1}, K_t^n \geq K_t^{n+1}.$$

Set  $Y_t = \sup_n Y_t^n$ ,  $K_t = \inf_n K_t^n$ . By remark 3.1, we have  $0 \leq \theta_t(\xi^n \vee a_T) \leq \theta_t(\xi \vee a_T) \leq \theta_0(\xi \vee a_T)$ , then

$$L_t \leq Y_t \leq \frac{1}{\gamma} \ln(E[\theta_0(\xi \vee a_T) | \mathcal{F}_t])$$

in view of the dominated convergence theorem and assumption 2.2. So  $Y \in \mathbf{S}^2(0, T)$  and  $K \in \mathbf{A}^2(0, T)$  since  $E[(K_T)^2] \leq E[(K_T^n)^2]$ .

Let us introduce the following stopping times

$$\tau_m = \inf\{t \in [0, T], \frac{1}{\gamma} \ln(E[\theta_t(\xi \vee a_T) | \mathcal{F}_t]) \geq m\} \wedge T.$$

Then denote  $(Y^{n,m}, Z^{n,m}, K^{n,m}) = (Y_{t \wedge \tau_m}^n, Z_t^n 1_{\{t < \tau_m\}}, K_{t \wedge \tau_m}^n)$ , which satisfy the following RB-SDE

$$\begin{aligned} Y_t^{n,m} &= \xi^{n,m} + \int_t^T 1_{\{s \leq \tau_m\}} f(s, Y_s^{n,m}, Z_s^{n,m}) ds + K_T^{n,m} - K_t^{n,m} - \int_t^T Z_s^{n,m} dB_s, \\ Y_t^{n,m} &\geq L_t, \int_0^T (Y_t^{n,m} - L_t) dK_t^{n,m} = 0, \end{aligned}$$

where  $\xi^{n,m} = Y_T^{n,m} = Y_{\tau_m}^n$ .

For  $m$  fixed, we have that  $\{\xi^{n,m}\}$  is increasing in  $n$ , and bounded by  $m$ , in view of  $\sup_n \sup_t |Y_t^{n,m}| \leq m$ . Now we apply the monotone stability theorem 4.1 to  $\{Y^{n,m}\}_{n \in \mathbb{N}}$ . Setting  $Y_t^m = \sup_n Y_t^{n,m}$ , then  $Y^{n,m}$  converge uniformly to  $Y^m$  on  $[0, T]$  and there exist processes  $Z^m \in \mathbf{H}_d^2(0, T)$  and  $K^m \in \mathbf{A}^2(0, T)$ , such that  $Z^{n,m} \rightarrow Z^m$  in  $\mathbf{H}_d^2(0, T)$ , and  $K^{n,m}$  converges uniformly decreasingly to  $K^m$ . Furthermore,  $(Y^m, Z^m, K^m)$  solves

$$\begin{aligned} Y_t^m &= \xi^m + \int_t^T 1_{\{s \leq \tau_m\}} f(s, Y_s^m, Z_s^m) ds + K_T^m - K_t^m - \int_t^T Z_s^m dB_s, \\ Y_t^m &\geq L_t, \int_0^T (Y_t^m - L_t) dK_t^m = 0, \end{aligned}$$

where  $\xi^m = \sup_n Y_{\tau_m}^n$ .

Since  $\tau_m \leq \tau_{m+1}$ , with the definition of  $(Y^m, Z^m, K^m)$ , we deduce that

$$Y_{t \wedge \tau_m} = Y_{t \wedge \tau_m}^{m+1} = Y_t^m, Z_t^{m+1} 1_{\{t \leq \tau_m\}} = Z_t^m, K_{t \wedge \tau_m} = K_{t \wedge \tau_m}^{m+1} = K_t^m.$$

Since  $Y^m$  and  $K^m$  are continuous, and  $P - a.s.$   $\tau_m = T$  for  $m$  large enough, so  $Y$  and  $K$  are continuous on  $[0, T]$ . We define  $Z$  on  $[0, T)$  by setting

$$Z_t = Z_t^1 1_{\{t \leq \tau_1\}} + \sum_{m \geq 2} Z_t^m 1_{(\tau_{m-1}, \tau_m]}(t),$$

so  $Z_t 1_{\{t \leq \tau_m\}} = Z_t^m 1_{\{t \leq \tau_m\}} = Z_t^m$  and the triplet  $(Y, Z, K)$  satisfies

$$Y_{t \wedge \tau_m} = Y_{\tau_m} + \int_{t \wedge \tau_m}^{\tau_m} f(s, Y_s, Z_s) ds + K_{\tau_m} - K_{t \wedge \tau_m} - \int_{t \wedge \tau_m}^{\tau_m} Z_s dB_s. \quad (5)$$

Since

$$\begin{aligned} P\left(\int_0^T |Z_s|^2 ds\right) &= P\left(\int_0^T |Z_s|^2 ds = \infty, \tau_m = T\right) + P\left(\int_0^T |Z_s|^2 ds = \infty, \tau_m < T\right) \\ &\leq P\left(\int_0^T |Z_s|^2 ds = \infty\right) + P(\tau_m < T), \end{aligned}$$

with  $\tau_m \nearrow T$ , as  $m \rightarrow \infty$ , we deduce that  $\int_0^T |Z_s|^2 ds < \infty$ ,  $P$ -a.s. Finally letting  $m \rightarrow \infty$  in (5), we get that  $(Y, Z, K)$  verifies the equation.

On the other hand,  $Y_t^m \geq L_t$ , so  $Y_t \geq L_t$  on  $[0, T]$  and for each  $m$ ,  $\int_0^T (Y_t^m - L_t) dK_t^m = 0$ , which implies  $\int_0^{\tau_m} (Y_t - L_t) dK_t = 0$ , for each  $m$ . Furthermore  $P$ -a.s. for  $m$  large enough,  $\tau_m = T$  so. we have  $\int_0^T (Y_t - L_t) dK_t = 0$ ,  $P$ -a.s..

To complete the proof, we need to prove that under the assumption 2.4 the process  $Z$  is in  $\mathbf{H}_d^2(0, T)$ .

If  $(Y, Z, K)$  is a solution of the RBSDE $(\xi, f, L)$  constructed as before, then

$$L_t \leq Y_t \leq \frac{1}{\gamma} \ln(E[\theta_t(|\xi| \vee a_T) | \mathcal{F}_t]), E[(K_T)^2] < +\infty. \quad (6)$$

So under the assumption 2.4, we get,

$$E\left[\sup_{0 \leq t \leq T} e^{2\gamma|Y_t|}\right] < +\infty. \quad (7)$$

For  $n \geq 1$ , let  $\sigma_n$  be the following stopping time:

$$\sigma_n = \inf\{t \geq 0, \int_0^t e^{2\gamma|Y_s|} |Z_s|^2 ds \geq n\} \wedge T,$$

and consider the following function

$$v(x) = \frac{1}{\gamma^2} (e^{\gamma x} - 1 - \gamma x).$$



By Itô's formula applied to  $v(|Y_t|)$ , with the notation

$$\text{sgn}(x) = \begin{cases} 1, & x > 0, \\ -1, & x \leq 0, \end{cases}$$

we get on  $[0, t \wedge \sigma_n]$ ,

$$\begin{aligned} v(|Y_0|) &= v(|Y_{t \wedge \sigma_n}|) + \int_0^{t \wedge \sigma_n} [v'(|Y_s|) \text{sgn}(Y_s) f(s, Y_s, Z_s) - \frac{1}{2} v''(|Y_s|) |Z_s|^2] ds \\ &\quad + \int_0^{t \wedge \sigma_n} v'(|Y_s|) \text{sgn}(Y_s) dK_s - \int_0^{t \wedge \sigma_n} v'(|Y_s|) \text{sgn}(Y_s) Z_s dB_s. \end{aligned}$$

From the assumption 2.1 and  $v'(x) \geq 0$ , for  $x > 0$ , we get

$$\begin{aligned} v(|Y_0|) &\leq v(|Y_{t \wedge \sigma_n}|) + \int_0^{t \wedge \sigma_n} v'(|Y_s|) (\alpha + \beta |Y_s|) ds + \sup_{0 \leq s \leq T} (v'(|Y_s|) \cdot K_T) \\ &\quad - \int_0^{t \wedge \sigma_n} v'(|Y_s|) \text{sgn}(Y_s) Z_s dB_s - \frac{1}{2} \int_0^{t \wedge \sigma_n} (v''(|Y_s|) - \gamma v'(|Y_s|)) |Z_s|^2 ds. \end{aligned} \quad (8)$$

Notice that  $(v'' - \gamma v')(x) = 1$ , for  $x \geq 0$ ; taking expectation in (8), we get

$$\begin{aligned} \frac{1}{2} E \int_0^{t \wedge \sigma_n} |Z_s|^2 ds &\leq E \left[ \frac{1}{\gamma^2} \sup_{0 \leq s \leq T} e^{\gamma |Y_t|} + \frac{1}{\gamma} \int_0^T e^{\gamma |Y_t|} (\alpha + \beta |Y_s|) ds \right] \\ &\quad + \frac{1}{\gamma} (E [\sup_{0 \leq s \leq T} e^{2\gamma |Y_s|}])^{\frac{1}{2}} \cdot (E[(K_T)^2])^{\frac{1}{2}}. \end{aligned} \quad (9)$$

By Fatou's lemma, with (6) and (7), letting  $n \rightarrow \infty$  in (9), we obtain  $E \int_0^T |Z_s|^2 ds < \infty$ .  $\square$

## 5 One extension

In this section, we extend our results to a more general case when the coefficient  $f$  is super-linear in  $y$ . Let  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a non-decreasing convex  $\mathcal{C}^1$  function with  $h(0) > 0$  such that

$$\int_0^{+\infty} \frac{du}{h(u)} = +\infty, \text{ and } \sup_{y>0} e^{-\gamma y} h(y) < +\infty \quad (10)$$

We assume:

**Assumption 2.5.** the coefficient  $f$  is continuous in  $(y, z)$  for  $t \in [0, T]$ , and there exists  $\gamma > 0$  such that for  $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$ ,

$$|f(t, y, z)| \leq h(|y|) + \frac{\gamma}{2} |z|^2.$$

Obviously, the linear increasing condition in assumption 2.1 corresponds to  $h(y) = \alpha + \beta y$ , but we can also give a superlinear growth in  $y$ , for example we can take  $h(y) = \alpha(y + e) \ln(y + e)$ .

Before giving our integrability condition for the terminal value  $\xi$ , we need some modifications. According to (10), we denote  $c_0 = \sup_{p \in (0,1)} \gamma p h(-\frac{\ln p}{\gamma})$  and

$$p_0 = \inf\{p \geq 1 : \gamma p h(\frac{\ln p}{\gamma}) \geq c_0\}.$$

Finally, we define

$$H(p) = \gamma p h(\frac{\ln p}{\gamma}) 1_{\{p \geq p_0\}} + c_0 1_{\{p < p_0\}}.$$

Then  $H$  is convex and we have the following lemma.

**Lemma 5.1.** *For  $x \in \mathbb{R}$ , the reflected BODE*

$$\begin{aligned} \theta_t(x) &= e^{\gamma x} + \int_t^T H(\theta_s(x)) ds + k_T(x) - k_t(x), \\ \theta_t(x) &\geq a_t, \int_0^T (\theta_t(x) - a_t) dk_t(x) = 0. \end{aligned}$$

has a unique solution  $(\theta_t(x), k_t(x))_{0 \leq t \leq T}$ . Moreover  $\theta_t(x)$  is decreasing on  $t$  and continuous increasing on  $x$ .

*Proof.* The results follows easily from the representation of the solution:

$$\begin{aligned} \theta_t(x) &= \max\{\varphi_t(T, e^{\gamma x}), \sup_{t \leq s < T} \varphi_t(s, e^{\gamma a_s})\} \\ &= \sup_{t \leq s \leq T} [\int_t^s H(\theta_r(x)) dr + e^{\gamma a_s} 1_{\{s < T\}} + e^{\gamma x} 1_{\{s = T\}}], \end{aligned}$$

where  $\varphi_t(s, e^{\gamma a_s})$  (resp.  $\varphi_t(T, e^{\gamma x})$ ) is a solution of ODE on  $[0, s]$  (resp.  $[0, T]$ ) associated to  $(e^{\gamma a_s}, H)$  (resp.  $(e^{\gamma x}, H)$ ), and the existence results about the non reflected ODE  $(e^{\gamma a_s}, H)$ , see lemma 6 in [2].  $\square$

Now we give our third integrability condition for the terminal condition  $\xi$ :

**Assumption 2.6.**  $\theta_0(\xi \vee a_T)$  is integrable.

Exactly as in the linear case, we can prove the following existence result:

**Theorem 5.1.** *Under assumptions 2.3, 2.5 and 2.6, the reflected BSDE associated to  $(\xi, f, L)$  has at least one solution  $(Y, Z, K)$  such that*

$$\begin{aligned} Y_t &= \xi + \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dB_s, \\ Y_t &\geq L_t, \int_0^T (Y_t - L_t) dK_t = 0. \end{aligned}$$

Moreover, we have  $L_t \leq Y_t \leq \frac{1}{\gamma} \ln(E[\theta_t(\xi \vee a_T) | \mathcal{F}_t])$ .

## 6 Appendix

### 6.1 Trajectory comparison theorem

In this subsection, we prove a trajectory comparison theorem for RBSDE's under a Lipschitz condition.

**Theorem 6.1.** *Suppose that for  $i = 1, 2$ ,  $\xi^i \in \mathbf{L}^2(\mathcal{F}_T)$ ,  $f^i(t, y, z)$  are Lipschitz functions in  $y$  and  $z$ , i.e. there exists a  $\mu > 0$ , such that for  $y_1, y_2 \in \mathbb{R}$ ,  $z_1, z_2 \in \mathbb{R}^d$ ,*

$$|f^i(t, y_1, z_1) - f^i(t, y_2, z_2)| \leq \mu(|y_1 - y_2| + |z_1 - z_2|),$$

*with  $f^i(t, 0, 0) \in \mathbf{H}^2(0, T)$ , and  $L^i$  are adapted continuous processes, with*

$$\xi^i \geq L_T^i \text{ and } E(\sup_t ((L_t^i)^+)^2) < +\infty.$$

*Let  $(Y^i, Z^i, K^i)$ ,  $i = 1, 2$ , be the solutions of the RBSDE's  $(\xi^i, f^i, L^i)$ , respectively. Moreover, we set  $\forall t \in [0, T]$ ,  $P$ -a.s.*

$$\xi^1 \leq \xi^2, f^1(t, Y_t^1, Z_t^1) \leq f^2(t, Y_t^1, Z_t^1), L_t^1 \leq L_t^2.$$

*Then  $Y_t^1 \leq Y_t^2$ , a.s., for  $t \in [0, T]$ .*

**Remark 6.1.** *We have the same result under the condition  $f^1(t, Y_t^2, Z_t^2) \leq f^2(t, Y_t^2, Z_t^2)$ .*

*Proof.* Applying Itô's formula to  $[(Y_t^1 - Y_t^2)^+]^2$ , then taking expectation, with Lipschitz condition, we get

$$E[(Y_t^1 - Y_t^2)^+]^2 \leq (2\mu^2 + 2\mu)E \int_t^T [(Y_s^1 - Y_s^2)^+]^2 ds.$$

From Gronwall's inequality, we deduce that  $(Y_t^1 - Y_t^2)^+ = 0$ ,  $t \in [0, T]$ , i.e.  $Y_t^1 \leq Y_t^2$ .  $\square$

### 6.2 Existence and uniqueness of a solution for reflected backward ODE's with one continuous barrier

We recall the definition of the space

$$\mathbf{L}_\gamma^2(\mathbb{R}) = \{ (v_t)_{0 \leq t \leq T} : [0, T] \rightarrow \mathbb{R}, \text{ s.t. } \int_0^T e^{\gamma t} |v_t|^2 dt < \infty \}, \text{ for } \gamma \in \mathbb{R}.$$

Consider the reflected backward ordinary differential equation(reflected BODE in short) reflected to one continuous barrier  $l$  on  $[0, T]$ , with terminal value  $x \in \mathbb{R}$ , whose solution is a couple  $(y_t, k_t)_{0 \leq t \leq T}$ , with  $y \in \mathbf{L}_0^2(\mathbb{R})$  is continuous, and  $k$  is a continuous increasing process,  $k_0 = 0$ , and the followings hold

$$\begin{aligned} y_t &= x + \int_t^T \phi(y_s) ds + k_T - k_t, \\ y_t &\geq l_t, \int_0^T (y_s - l_s) dk_s = 0. \end{aligned} \tag{11}$$

Here we suppose

**Assumption A1.** the function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$ , is continuous, and there exists a strictly positive function  $l_0$ , such that  $|\phi(y)| \leq l_0(y)$ , with  $\int_0^\infty \frac{dy}{l_0(y)} = \infty$ . And  $\phi$  is increasing in  $y$ .

**Assumption A2.** the barrier  $l$  satisfies:  $\alpha \leq l_t \leq \beta$ , with  $\beta > 1$ ,  $0 < \alpha \leq 1$ . And  $l_T \leq x$ .

Furthermore we assume that

**Assumption A3.** the non reflected BODE's with any terminal value  $x$ , any terminal time  $0 \leq s \leq T$ , and the coefficient  $\phi$ , have a unique solution.

Our main result is the

**Theorem 6.2.** *Under assumptions A1, A2 and A3, the reflected BODE (11) admits one unique solution  $(y_t, k_t)_{0 \leq t \leq T}$ . Moreover*

$$y_t = \sup_{t \leq s \leq T} u_t^s = \sup_{t \leq s \leq T} \left[ \int_t^s \phi(y_r) dr + x 1_{\{s=T\}} + l_s 1_{\{s < T\}} \right],$$

where  $(u_t^s)_{0 \leq t \leq s}$  is the unique solution of the following ODE defined on  $[0, s]$

$$u_t^s = (x 1_{\{s=T\}} + l_s 1_{\{s < T\}}) + \int_t^s \phi(u_r^s) dr.$$

**Remark 6.2.** *The solution  $y$  is the smallest process which satisfies the equation and  $y_t \geq l_t$ ,  $t \in [0, T]$ , i.e. if another couple  $(y', k')$  satisfies aussi the equation and  $y'_t \geq l_t$ , then  $y_t \leq y'_t$ . But the increasing process  $k$  is not the smallest one.*

We first consider the existence of a solution.

### 6.2.1 Existence

For the existence, we do not need the monotonicity condition of  $\phi$  in  $y$  in assumption A1 and assumption A3. The proof is done in three steps:

- a)  $\phi$  is Lipschitz in  $y$ ,
- b)  $\phi$  is linear increasing in  $y$ ,
- c)  $\phi$  is superlinear increasing in  $y$ .

We consider first

**a)** the case  $\phi$  Lipschitz in  $y$ , i.e. there exists a constant  $\mu \in \mathbb{R}$ , such that for  $y, y' \in \mathbb{R}$ ,  $|\phi(y) - \phi(y')| \leq \mu |y - y'|$ .

When  $\phi = \phi_t$  in  $\mathbf{L}_0^2(\mathbb{R})$ , which means  $\phi$  does not depend of  $y$ , it is easy to check that the solution of such an equation is  $y_t = \max\{x + \int_t^T \phi_s ds, l_t\}$ ,  $k_t = \int_0^t (l_s - (x + \int_s^T \phi_r dr))^+ ds$ . Thanks to the Lipschitz property of  $\phi$  we can construct a strict contraction in  $\mathbf{L}_\gamma^2(\mathbb{R})$ , beginning with a given process  $\{y^1\} \in \mathbf{L}_0^2(\mathbb{R})$ . So the reflected BODE admits one unique solution.  $\square$

Moreover, we have a comparison theorem:

**Theorem 6.3.** *We consider the equations associated to  $(x^i, \phi^i, l)$ ,  $i = 1, 2$ , and assume that  $\phi^1$  and  $\phi^2$  satisfy the Lipschitz assumptions. Let  $(y^i, k^i)$  be the respective solutions of these equations. Moreover, we assume for  $t \in [0, T]$ ,*

$$x^1 \geq x^2, \phi^1(y_t^1) \geq \phi^2(y_t^1), l_t^1 \geq l_t^2.$$

Then  $y_t^1 \geq y_t^2$ .

**Proof.** We consider  $((y_t^2 - y_t^1)^+)^2$ . Notice that on the set  $\{y_t^2 \geq y_t^1\}$ ,  $y_t^2 > y_t^1 \geq l_t^1 \geq l_t^2$ , so we have

$$\begin{aligned} & \int_t^T (y_t^2 - y_t^1)^+ d(k_s^2 - k_s^1) \\ & \leq \int_t^T (y_t^2 - l_t^2) dk_s^2 - \int_t^T (y_t^1 - l_t^2) 1_{\{y_t^2 > y_t^1\}} dk_s^2 - \int_t^T (y_t^2 - y_t^1)^+ dk_s^1 \leq 0. \end{aligned}$$

Consequently, we get

$$((y_t^2 - y_t^1)^+)^2 \leq 2k \int_t^T ((y_s^2 - y_s^1)^+)^2 ds.$$

It follows immediately that  $(y_t^2 - y_t^1)^+ = 0$ , i.e.  $y_t^1 \geq y_t^2$ .  $\square$

**Remark 6.3.** The result is still true under the assumption  $\phi^1(y_t^2) \geq \phi^2(y_t^2)$ ,  $t \in [0, T]$ .

**b)** We now suppose that  $\phi$  is continuous and linear increasing in  $y$ , i.e. there exists a constant  $\mu_l \in \mathbb{R}$ , such that for  $y \in \mathbb{R}$ ,  $|\phi(t, y)| \leq \mu_l(1 + |y|)$ .

**Lemma 6.1.** Under the assumptions **b)** and A2, there exists a minimal solution  $(y_t, k_t)_{0 \leq t \leq T}$  of the reflected BODE( $x, \phi, l$ ).

*Proof.* We consider the following approximation: for  $n \in \mathbb{N}$ , define

$$\phi_n(y) = \inf_{x \in \mathbf{Q}} \{\phi(x) + n|y - x|\}, \quad (12)$$

then for  $n \geq \mu_l$ ,  $\phi_n$  satisfies

- 1) Linear increasing:  $|\phi_n(y)| \leq \mu_l(1 + |y|)$ ;
  - 2) Monotonicity:  $\phi_n(y) \nearrow \phi(y)$ ;
  - 3) Lipschitz condition:  $|\phi_n(y) - \phi_n(y')| \leq n|y - y'|$ ;
  - 4) Strong convergence: If  $y_n \rightarrow y$ , then  $\phi_n(y_n) \rightarrow \phi(y)$ , as  $n \rightarrow \infty$ .
- (13)

By the result of a), for each  $n \in \mathbb{N}$ , there exists a unique solution  $(y^n, k^n)$  of the equation  $(x, \phi_n, l)$ . It's easy to check that the solutions  $(y^n)$  are bounded uniformly in  $n$ , i.e.  $\sup_{0 \leq t \leq T} (y_t^n)^2 \leq C$ . Thanks to the comparison theorem 6.3, and 2) of (13), we know that  $y_t^n \nearrow y_t$ , for  $t \in [0, T]$ . By Fatou's lemma, we get  $\sup_{0 \leq t \leq T} (y_t)^2 \leq C$ , and  $\int_0^T |y_t^n - y_t|^2 ds \rightarrow 0$ , in view of the dominated convergence theorem.

Then we prove that the convergence still holds in some stronger sense; for  $n, p \in \mathbb{N}$ , we have

$$\sup_{0 \leq t \leq T} (y_t^n - y_t^p)^2 \leq 2 \left( \int_0^T (y_s^n - y_s^p)^2 ds \right)^{\frac{1}{2}} \left( \int_0^T (\phi_n(y_s^n) - \phi_p(y_s^p))^2 ds \right)^{\frac{1}{2}}.$$

By 1) of (13) and the estimate of  $(y^n)$ , we get easily  $\int_0^T (\phi_n(y_s^n) - \phi_p(y_s^p))^2 ds \leq C$ , so  $\sup_{0 \leq t \leq T} (y_t^n - y_t^p)^2 \rightarrow 0$ , as  $n, p \rightarrow \infty$  and the limit  $y$  is continuous.

For  $\{k^n\}$ , it is easy to check that  $\sup_{0 \leq t \leq T} (k_t^n - k_t^p)^2 \rightarrow 0$ , as  $n, p \rightarrow \infty$ . Then there exists a increasing continuous process  $k$ , such that  $(y, k)$  satisfies the equation. At last, we consider

$$\int_0^T (y_t^n - l_t) dk_t^n - \int_0^T (y_t - l_t) dk_t \leq \int_0^T (y_t - l_t) d(k_t^n - k_t) \rightarrow 0,$$

in view of  $\sup_{0 \leq t \leq T} (k_t^n - k_t)^2 \rightarrow 0$ , as  $n \rightarrow \infty$ . Since  $y_t^n \geq l_t$ , we get  $y_t \geq l_t$ , for  $t \in [0, T]$ . With  $\int_0^T (y_t^n - l_t) dk_t^n = 0$ , we have  $\int_0^T (y_t - l_t) dk_t = 0$ . The proof is complete.  $\square$

For the maximal solution, it is sufficient to replace (12) by

$$\phi_n(y) = \sup_{x \in \mathbf{Q}} \{\phi(x) - n|y - x|\},$$

which is a sequence of Lipschitz functions which converge decreasingly to  $\phi$ . Then using the same approximation method as before, we obtain the existence of the maximal solution. We have also the following comparison theorem.

**Theorem 6.4.** *Let us consider  $\phi_1, \phi_2$  which satisfy the condition **b**). We suppose that for  $y \in \mathbb{R}$ ,  $t \in [0, T]$ ,*

$$x^1 \geq x^2, \phi_1(y) \geq \phi_2(y), l_t^1 \geq l_t^2.$$

*For the maximal (minimal) solution  $(y^i, k^i)$ ,  $i = 1, 2$ , of the reflected equation associated to  $(x^i, \phi^i, l)$ , we have  $y_t^1 \geq y_t^2$ , for  $t \in [0, T]$ .*

*Proof.* The result comes easily from the approximation and theorem 6.3.  $\square$

**c)** We consider finally the case  $\phi$  is continuous and superlinear in  $y$ , i.e.  $|\phi(y)| \leq l_0(y)$ , with  $\int_0^\infty \frac{dy}{l_0(y)} = \infty$ .

Let  $v_t$  be the solution of the ordinary differential equation:  $v_t = b + \int_t^T l_0(v_s) ds$ , where  $b = x \vee \sup_t l_t$  (see [7] Lemma 1). Then we have

**Lemma 6.2.** *Under the assumption **c**), the reflected equation has a maximal solution  $(y, k)$ , which satisfies:  $\underline{m} \leq l_t \leq y_t \leq v_t \leq v_0$  where  $\underline{m} := \inf_t l_t$ .*

*Proof.* Let  $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a smooth function such that

$$\rho(x) = \begin{cases} \frac{r}{2}, & 0 < x < \frac{r}{2}; \\ x, & r \leq x \leq R; \\ 2R, & x > 2R. \end{cases}$$

Here  $r$  and  $R$  are two real number such that  $0 < r < \underline{m}$  and  $R > v_0$ . It is a direct result that the unique solution of the equation

$$v_t^\rho = b + \int_t^T l_0(\rho(v_s^\rho)) ds$$

satisfies  $v_t^\rho = v_t$  and  $v_t^\rho = v_t \geq v_T \geq l_t$  for  $t \in [0, T]$ . So  $(v^\rho, 0)$  can be considered as the solution of the reflected BODE associated to  $(b, l_0(y), l)$ . Then we consider the following

reflected BODE with one barrier  $l$ :

$$\begin{aligned} y_t^\rho &= x + \int_t^T \phi(\rho(y_s^\rho)) ds + k_T^\rho - k_t^\rho, \\ y_t^\rho &\geq l_t, \int_0^T (y_t^\rho - l_t) dk_t^\rho = 0. \end{aligned}$$

Since  $\phi(\rho(y))$  is bounded and continuous, this equation admits a maximal solution  $(y^\rho, k^\rho)$ . Thanks to the comparison theorem, we get  $y_t^\rho \leq v_t^\rho \leq v_0 < R$ . With  $y_t^\rho \geq l_t \geq \underline{m} > r$ , it follows that

$$\phi(\rho(y_s^\rho)) = \phi(y_s^\rho),$$

i.e.  $(y^\rho, k^\rho)$  is also a maximal solution of the reflected BODE associated to  $(x, \phi, l)$ .  $\square$

We have still a comparison theorem, which follows easily from the proof of existence and theorem 6.4.

**Theorem 6.5.** *Consider functions  $\phi_1, \phi_2$  which satisfy condition **c**). We suppose, for  $(t, y) \in [0, T] \times \mathbb{R}$ .*

$$x^1 \geq x^2, \phi_1(y) \geq \phi_2(y), l_t^1 \geq l_t^2,$$

*For the maximal (minimal) solutions  $(y^i, k^i)$ ,  $i = 1, 2$ , of the reflected equations associated to  $(x^i, \phi^i, l)$ , we have  $y_t^1 \geq y_t^2$ , for  $t \in [0, T]$ .*

## 6.2.2 Uniqueness and characterization of the solution

Here we will give a characterization of the solution of the reflected BODE under assumptions A1, A2 and A3. First, we consider the following lemma:

**Lemma 6.3.** *Let  $u^\varepsilon$  be the unique solution of the following BODE for some  $\varepsilon > 0$ ,  $\varepsilon \in \mathbb{R}$ ,*

$$u_t^\varepsilon = x - \varepsilon + \int_t^T \phi(u_s^\varepsilon) ds, \quad (14)$$

*Then  $u_t^\varepsilon$  converge increasing to  $u_t$  as  $\varepsilon \rightarrow 0$ , where  $u_t$  is the solution of the BODE  $u_t = x + \int_t^T \phi(u_s) ds$ .*

*Proof.* By comparison theorem 6.5, we know that  $u_t^{\varepsilon_1} \geq u_t^{\varepsilon_2}$ , for  $\varepsilon_1 \leq \varepsilon_2$ . So  $u_t^\varepsilon \nearrow u_t$ , for  $t \in [0, T]$ , as  $\varepsilon \rightarrow 0$ . Then the result follows easily from the continuity of  $\phi$  in  $y$  and the boundedness of  $u_t^\varepsilon$ .  $\square$

Now we prove a useful inequality.

**Lemma 6.4.** *Let  $u_t$  be the solution of BODE  $u_t = x + \int_t^T \phi(u_s) ds$ , and  $y_t$  which satisfies  $y_t \geq x + \int_t^T \phi(y_s) ds$ , on  $[0, T]$ . Then  $u_t \leq y_t$ , for  $t \in [0, T]$ .*

*Proof.* For any  $\varepsilon > 0$ ,  $y_T \geq x > x - \varepsilon = u_T^\varepsilon$ , where  $u_t^\varepsilon$  is the solution of (14). Suppose that there exists a  $\bar{\tau}$ , such that  $u_{\bar{\tau}}^\varepsilon = y_{\bar{\tau}}$  and  $y_s > u_s^\varepsilon$  on  $[\bar{\tau}, T]$ . It follows from the monotonicity of  $\phi$  on  $y$  that

$$y_{\bar{\tau}} \geq x + \int_{\bar{\tau}}^T \phi(y_s) ds \geq x + \int_{\bar{\tau}}^T \phi(u_s^\varepsilon) ds > u_{\bar{\tau}}^\varepsilon,$$

which is a contradiction. So  $y_t > u_t^\varepsilon$  on  $[0, T]$ , for any  $\varepsilon > 0$ . Let  $\varepsilon \rightarrow 0$ , with lemma 6.3, we have  $u_t \leq y_t$ , on  $[0, T]$ .  $\square$

With the help of these Lemmas, we give the representation of the solution of the reflected BODE.

**Proposition 6.1.** *Under the assumptions A1, A2 and A3, assume that  $(y_t, k_t)_{0 \leq t \leq T}$  is a solution of the following reflected BODE*

$$\begin{aligned} y_t &= x + \int_t^T \phi(y_s) ds + k_T - k_t, \\ y_t &\geq l_t, \int_0^T (y_s - l_s) dk_s = 0. \end{aligned} \tag{15}$$

Then we have for  $t \in [0, T]$ ,

$$y_t = \sup_{t \leq s \leq T} u_t^s = \sup_{t \leq s \leq T} \left[ \int_t^s \phi(y_r) dr + x 1_{\{s=T\}} + l_s 1_{\{s < T\}} \right],$$

where  $(u_t^s)_{0 \leq t \leq s}$  is the solution of the BODE defined on  $[0, s]$  with coefficient  $\phi$  and terminal value  $x 1_{\{s=T\}} + l_s 1_{\{s < T\}}$ .

*Proof.* For  $t \in [0, T]$ , since  $y$  is a solution of the reflected BODE, with lemma 6.4, we get  $y_t \geq u_t^s$ , for  $s \in [t, T]$ . Denote

$$D_t = \inf\{u \in [t, T], y_u = l_u\} \wedge T.$$

Notice that  $k$  is an increasing process and  $\int_0^T (y_s - l_s) dk_s = 0$ , then  $k_{D_t} = k_t$ . It follows that

$$y_t = u_t^{D_t},$$

which implies the first equality. For the second one, from (15), it follows

$$\begin{aligned} y_t &= y_s + \int_t^s \phi(y_r) dr + k_T - k_s \\ &\geq \int_t^s \phi(y_r) dr + x 1_{\{s=T\}} + l_s 1_{\{s < T\}}. \end{aligned}$$

With the same  $D_t$ , we have

$$y_t = \int_t^{D_t} \phi(y_r) dr + x 1_{\{D_t=T\}} + l_{D_t} 1_{\{D_t < T\}}.$$

The proof is complete.  $\square$

**Remark 6.4.** *The function  $H(p) = p(\alpha\gamma + \beta \ln p 1_{[1, +\infty[}(p)) + \alpha\gamma 1_{(-\infty, 1)}(p)$  satisfies assumption A3, the existence and uniqueness of the solution of non reflected BODE's (see [2]). Consequently we have the result of the theorem 6.2 relatively to  $H$ .*



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